Testing quantum circuits and detecting insecure encryption

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Overview

We show that computational problem of testing the behaviour of quantum circuits is hard for QMA. This generalizes previous techniques to prove QMA-hardness for circuit problems. We apply this result to show the hardness of a weak version of detecting the insecurity of a symmetric-key quantum encryption system, or alternately the problem of determining when a quantum channel is not private. We also give a QMA protocol for this problem to show that it is QMA-complete.

Testing quantum circuits

Given a circuit C, does this circuit act like some known circuit C_0 on a large subspace of inputs, or does it act like some other known circuit C_1 on the whole input space? We show this problem is hard for any two families of quantum circuits that are not too close.

Problem ($\mathrm{CT}(\varepsilon, \delta, \mathcal{C}_0, \mathcal{C}_1)$)**.** Let $0 < \varepsilon < 1$, $0 < \delta \leq 1$, and $\mathcal{C}_0, \mathcal{C}_1$ be two uniform families of quantum circuits. The input is a circuit $C \in \mathbf{T}(\mathcal{X}, \mathcal{Y})$. Let C_0 and C_1 be the circuits from $\mathcal{C}_0$ and $\mathcal{C}_1$ that have the same input/output spaces as C. The problem is to decide:

Yes: there is a subspace S of $\mathcal{X}$ with $\dim S \geq (\dim \mathcal{X})^{1-\delta}$ such that for any $\mathcal{R}, \rho \in \mathbf{D}(S \otimes \mathcal{R})$

$$\|(C \otimes \mathbb{1}_{\mathcal{R}})(\rho) - (C_0 \otimes \mathbb{1}_{\mathcal{R}})(\rho)\|_{\mathrm{tr}} \leq \varepsilon,$$

No: $\|C - C_1\|_\diamond \leq \varepsilon$, i.e. for all $\mathcal{R}$ and any $\rho \in \mathbf{D}(\mathcal{X} \otimes \mathcal{R})$

$$\|(C \otimes \mathbb{1}_{\mathcal{R}})(\rho) - (C_1 \otimes \mathbb{1}_{\mathcal{R}})(\rho)\|_{\mathrm{tr}} \leq \varepsilon.$$

This problem is well-defined only for families $\mathcal{C}_0$ and $\mathcal{C}_1$ that do not violate the promise, i.e. any circuits whose output is not too close together. These are the C_0 and C_1 such that there does not exist a subspace T of $\mathcal{X}$ of size $\dim T > \dim \mathcal{X}^\delta$ such that for any input states $\rho \in \mathbf{D}(T \otimes \mathcal{R})$ we have

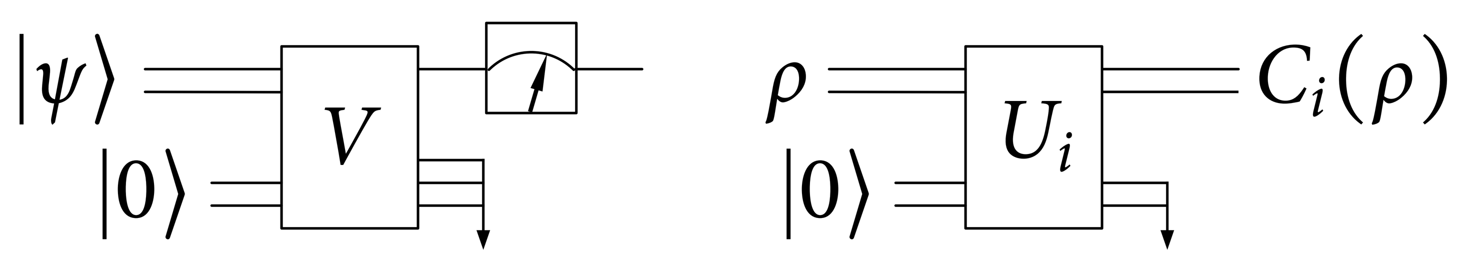
$$\|(C_0 \otimes \mathbb{1}_{\mathcal{R}})(\rho) - (C_1 \otimes \mathbb{1}_{\mathcal{R}})(\rho)\|_{\mathrm{tr}} \leq 2\varepsilon. \quad (1)$$

Note that $\mathcal{C}_0, \mathcal{C}_1$ are part of the problem definition: an algorithm to solve the problem may depend non-uniformly on these families. Notice also than when $\delta = 1$ the problem asks if there are *any* inputs ρ for which $C(\rho) \approx C_0(\rho)$ or if $C(\rho) \approx C_1(\rho)$ for all ρ.

QMA Hardness of circuit testing

Theorem. *$\mathrm{CT}(\varepsilon, \delta, \mathcal{C}_0, \mathcal{C}_1)$ is QMA-hard for any $0 < \varepsilon < 1$ such that $\varepsilon \geq 2^{-p}$ for some polynomial p, any constant $0 < \delta \leq 1$, and any uniform circuit families $\mathcal{C}_0, \mathcal{C}_1$ satisfying* (1).

Proof Sketch: We reduce an arbitrary (promise) problem in QMA to $\mathrm{CT}(\varepsilon, \delta, \mathcal{C}_0, \mathcal{C}_1)$. To prove that CT is QMA-hard, we embed the problem of deciding if an arbitrary QMA verifier V accepts into an equivalent instance of the CT problem.



(a) QMA verifier (b) Circuit C_i

Figure 1: The starting point for the reduction is a QMA verifier. The circuits implementing C_0 and C_1 are part of the problem definition and do not depend on V.

Using these circuits, the reduction constructs a circuit C that runs the verifier V and then behaves like either C_0 or C_1 depending on whether the verifier would have accepted part of the input state. C is a "yes" instance of CT if and only if V can be made to accept.

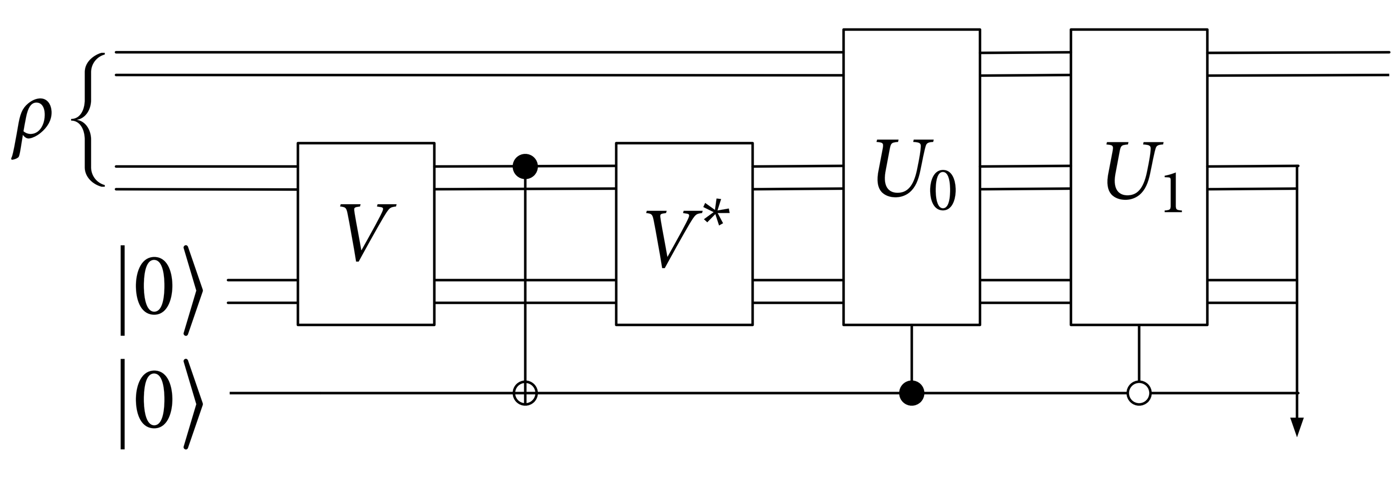


Figure 2: Instance C of the CT problem produced by the reduction.

Essential to the argument is that if the verifier V accepts or rejects with high probability, the result is essentially a "gentle measurement" of the output qubit. The portion of the input that is sent only to the circuits U_0 and U_1 serves to ensure that if V accepts any state, then V must "accept" on a subspace of dimension at least $(\dim \mathcal{X})^{1-\delta}$. Using this we show that the instance C is equivalent to deciding if V accepts some input state. □

Other QMA hardness results

The hardness of many circuit problems follows immediately from the hardness of the circuit testing problem, which can be used as a general tool to prove QMA-hardness. What follows is a list of some of these problem. Ω is the completely depolarizing channel.

Problem ((Mixed) Non-identity Check **(See [2])**)**.** Let $0 < \varepsilon < 1$. On input a circuit $C \in \mathbf{T}(\mathcal{X}, \mathcal{X})$, the promise problem is to decide between:

Yes: $\|C - \mathbb{1}\|_\diamond \geq 2 - \varepsilon$ and there exists an efficient unitary U such that on some pure state $|\psi\rangle \in \mathcal{X}$ we have $\|C(|\psi\rangle\langle\psi|) - U|\psi\rangle\langle\psi|U^*\|_{\mathrm{tr}} \leq \varepsilon$ and $\|U|\psi\rangle\langle\psi|U^* - |\psi\rangle\langle\psi|\|_{\mathrm{tr}} \geq 2 - \varepsilon$.

No: $\|C - \mathbb{1}\|_\diamond \leq \varepsilon$.

This is QMA-hard as $\mathrm{CT}(\varepsilon, 1, \mathcal{U}, \mathbb{1})$ is a special case for $\mathcal{U}$ is any uniform family of unitary quantum circuits that are not close to the identity. The requirement on yes instances that C is close to a unitary U on some input state is not needed for hardness, but is required for the phase-estimation based QMA verifier for this problem [2].

Problem (Non-isometry [3])**.** Let $0 < \varepsilon < 1/2$. On input a circuit $C \in \mathbf{T}(\mathcal{X}, \mathcal{Y})$ the promise problem is to decide between:

Yes: There exists $|\psi\rangle \in \mathcal{X}$ such that $\|(\Phi \otimes \mathbb{1}_{\mathcal{X}})(|\psi\rangle\langle\psi|)\|_\infty \leq \varepsilon$,

No: For all $|\psi\rangle \in \mathcal{X}$, $\|(\Phi \otimes \mathbb{1}_{\mathcal{X}})(|\psi\rangle\langle\psi|)\|_\infty \geq 1 - \varepsilon$.

This is QMA-hard as $\mathrm{CT}(\varepsilon, 1, \Omega, \mathbb{1})$ is a special case.

Problem (Pure Fixed Point)**.** Let $0 < \varepsilon < 1$. On input a circuit $C \in \mathbf{T}(\mathcal{X}, \mathcal{X})$ the promise problem is to decide between:

Yes: There exists $|\psi\rangle \in \mathcal{X}$ such that $\|C(|\psi\rangle\langle\psi|) - |\psi\rangle\langle\psi|\|_{\mathrm{tr}} \leq \varepsilon$

No: For any $|\psi\rangle \in \mathcal{X}$, $\|C(|\psi\rangle\langle\psi|) - |\psi\rangle\langle\psi|\|_{\mathrm{tr}} \geq 2 - \varepsilon$

This is QMA-hard as $\mathrm{CT}(\varepsilon, 1, \mathbb{1}, \Omega)$ is a special case.

Let $S_{\min}(C) = \min_\rho S(C(\rho))$ be the minimum output entropy of the channel C (where S is the von Neumann entropy). This problem is related to a problem in [1].

Problem (Minimum Output Entropy)**.** Let $0 < \varepsilon < 1/2$. On input a circuit $C \in \mathbf{T}(\mathcal{X}, \mathcal{X})$ the promise problem is to decide between:

Yes: $S_{\min}(C) \leq \varepsilon \log \dim \mathcal{X}$

No: $S_{\min}(C) \geq (1 - \varepsilon) \log \dim \mathcal{X}$

This is QMA-hard as $\mathrm{CT}(\varepsilon/2, 1, \mathbb{1}, \Omega)$ is a special case, by the Fannes Inequality.

Detecting insecure encryption

How difficult is it to verify the security of a symmetric-key quantum encryption scheme that acts on n qubits, given a full circuit implementation? The QMA-hardness of this problem implies that you cannot verify an encryption system from an untrusted party.

Problem (Detecting Insecure Encryption)**.** For $0 < \varepsilon < 1$ and $0 < \delta \leq 1$ an instance of the problem consists of a quantum circuit E that takes as input a quantum state as well as a m classical bits, such that for each $k \in \{0,1\}^m$ the circuit implements a quantum channel $E_k \in \mathbf{T}(\mathcal{X}, \mathcal{Y})$ with $\dim \mathcal{Y} \geq \dim \mathcal{X}$. The promise problem is to decide between:

Yes: There exists a subspace S of $\mathcal{X}$ with $\dim S \geq \dim \mathcal{X}^{1-\delta}$ such that for any reference space $\mathcal{R}$, any $\rho \in \mathbf{D}(S \otimes \mathcal{R})$, and any key k, $\|(E_k \otimes \mathbb{1}_{\mathcal{R}})(\rho) - \rho\|_{\mathrm{tr}} \leq \varepsilon$.

No: E is an ε-private channel, i.e. $\left\|\Omega - \frac{1}{2^m}\sum_{k \in \{0,1\}^m} E_k\right\|_\diamond \leq \varepsilon$, where Ω is the completely depolarizing channel in $\mathbf{T}(\mathcal{X}, \mathcal{Y})$, and there exists a polynomial-size quantum circuit D such that for all k we have $\|D_k \circ E_k - \mathbb{1}_{\mathcal{X}}\|_\diamond \leq \varepsilon$.

Informally, the problem is to distinguish two cases: either the circuit fails to encrypt a large subspace of the input (for all keys), or the channel is close to perfect.

The QMA-hardness of this problem follows from the hardness of CT. A QMA verifier can be constructed for this problem using the swap test.

Theorem. *For $0 < \varepsilon < 1/8$ and $0 < \delta \leq 1$, the problem $\mathrm{DI}_{\varepsilon,\delta}$ is QMA-complete.*

References

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[3] B. Rosgen. Testing non-isometry is QMA-complete. In *Proc. TQC*, pp. 63–76. 2010.