The Overlap Number of a Graph

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Outline

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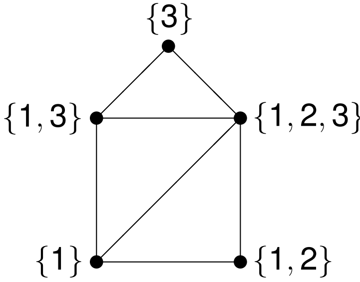
What is a Set Representation?

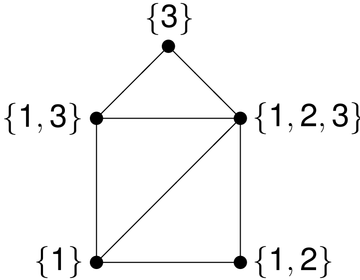
- An assignment of sets to the vertices of a graph is a *set representation* if two vertices are adjacent exactly when their sets satisfy some relation



Size of a Representation

- For $G = (V, E)$, we denote such a representation by $\mathcal{C} = \{S_v : v \in V\}$
- The *size* of a representation, $\mathcal{C}$, is

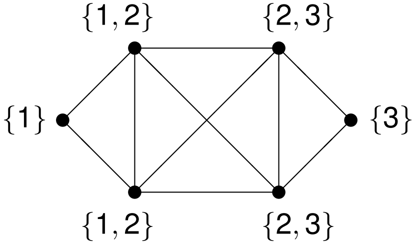
$$\left| \bigcup_{v \in V} S_v \right|$$



Outline

Intersection Representations

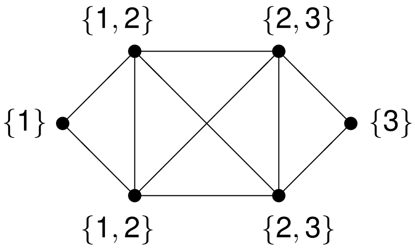
- An *intersection representation* is a set representation where u and v are adjacent if and only if $S_u \cap S_v \neq \emptyset$



Definition (Intersection Number)

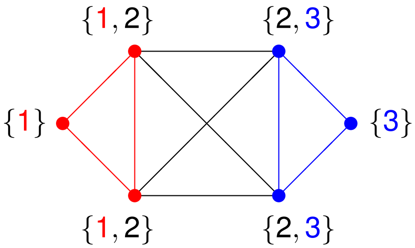
$\theta_e(G)$ is the size of a minimum intersection representation

Edge Clique Covers



- Intersection representations are equivalent to edge clique covers

Edge Clique Covers

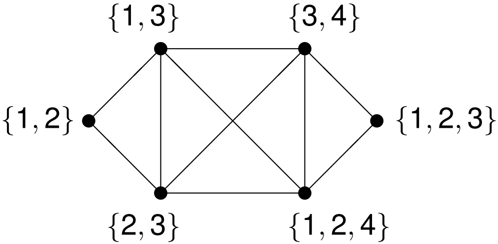


- Intersection representations are equivalent to edge clique covers
- This is used in many results on the intersection number

Outline

Overlap Representations

- In an *overlap representation*, u and v are adjacent if and only if S_u and S_v intersect and neither set is contained in the other

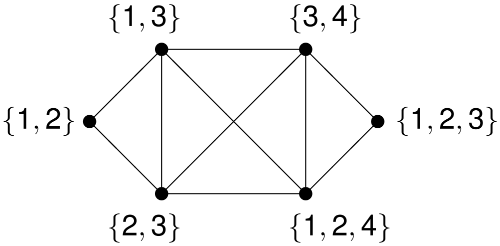


Definition (Overlap Number)

$\varphi(G)$ is the size of a minimum overlap representation

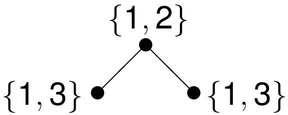
Disjointedness and Containment

- There are graphs for which disjointedness must be used in a minimum overlap representation



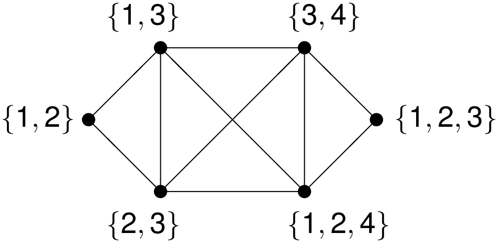
Disjointedness and Containment

- There are graphs for which disjointedness must be used in a minimum overlap representation
- There are also graphs that require containment



Cocomparability Graph Cover?

- Unlike intersection representations, the elements of an overlap representation induce cocomparability graphs



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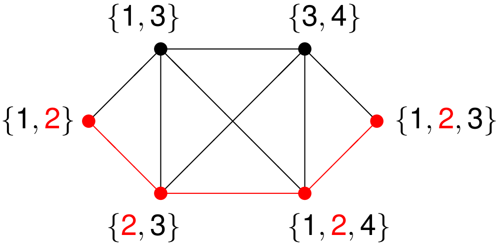

- This seems to rule out a simple method to find a representation

Upper Bounds

- Any intersection representation can be made into an overlap representation by adding a new element to each set.

Class	**Bound**
All Graphs[1]	$\varphi(G) \leq \lfloor (n^2 + 4n)/4 \rfloor$
Chordal[2]	$\varphi(G) \leq 2n$
Planar[3]	$\varphi(G) \leq \frac{10}{3}n - 6$
Trees	$\varphi(G) \leq n + 1$
Cocomparability	$\varphi(G) \leq n + 1$

[1] Erdős, Goodman, and Pósa, 1966
[2] Fulkerson and Gross, 1965
[3] Prisner, 1992

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Cliques

- $\varphi(K_n)$ is the number of elements required to construct n pairwise overlapping sets
- Milner's Theorem of 1966 gives exactly this quantity

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$$\varphi(K_n) = \left\{ \min m : n \leq \binom{m}{\lfloor \frac{m+2}{2} \rfloor} \right\}$$

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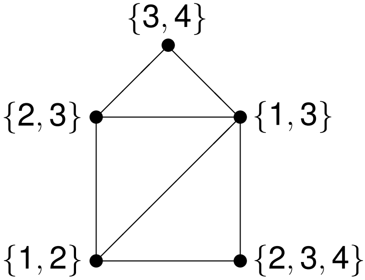
$$\varphi(K_n) = \left\{ \min m : n \leq \binom{m}{\lfloor \frac{m+2}{2} \rfloor} \right\}$$

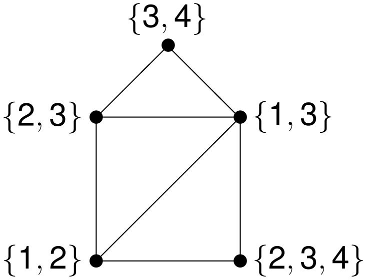
- This implies that $\varphi(K_n) \in \Theta(\log n)$
- In contrast, $\theta_e(K_n) = 1$

Complete *k*-partite Graphs



- Vertices do not need to be assigned distinct sets

Complete k-partite Graphs



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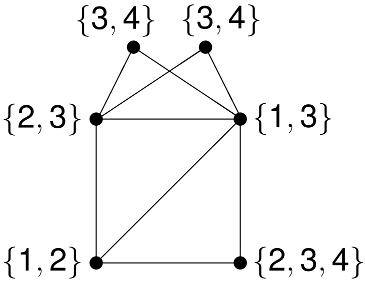

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Theorem

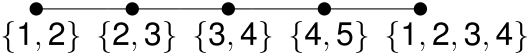
If G is a complete k-partite graph then $\varphi(G) = \varphi(K_k)$

Paths, Cycles, and Caterpillars

Theorem

For $n \geq 3$,

$$\varphi(P_n) = n$$



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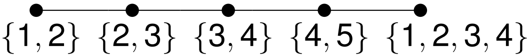

Corollary

Where T is a caterpillar with spine of length k and $n \geq 4$,

$$\begin{aligned} \varphi(C_n) &= n - 1 \\ \varphi(T) &= k + 2 \end{aligned}$$

Disconnected Graphs

Theorem

If G is a graph with connected components $B_1, B_2, \ldots, B_k$

$$\varphi(G) = \sum_{i=1}^{k} \varphi(B_i) - (k-1)$$

Disconnected Graphs

Theorem

If G is a graph with connected components $B_1, B_2, \ldots, B_k$

$$\varphi(G) = \sum_{i=1}^{k} \varphi(B_i) - (k-1)$$

- The elements of one component can play the role of a single element in the rest of the graph
- In an intersection representation, different components are forced to have disjoint representations

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Computing the Size of a Minimum Representation

- Computing $\theta_e(G)$ is **NP**-complete
- Reduction is from (Vertex) Clique Cover to Edge Clique Cover (Kou, Stockmeyer, Wong, 1978)

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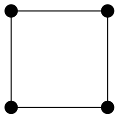
- Computing $\theta_e(G)$ is **NP**-complete
- Reduction is from (Vertex) Clique Cover to Edge Clique Cover (Kou, Stockmeyer, Wong, 1978)
- The complexity of computing $\varphi(G)$ is unknown

Containment Free Overlap Representations

- With containment forbidden in the representation, computing the overlap number of an arbitrary graph is **NP**-complete
- With no containment, any two sets that intersect must also overlap, so this is also an intersection number problem



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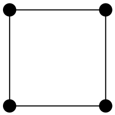

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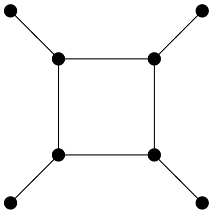

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- This can be extended to at most a constant number of containments between sets of the representation
- If no more than k containments are allowed, make $2k + 1$ copies of the input graph
- A many-one reduction can be found by adding extra components to the graph where a minimum representation is forced to “spend” all k containment relationships

Extending a Representation

- The basic problem is, given a graph, a set representation, and a vertex to add, can the representation be extended to include the new vertex?

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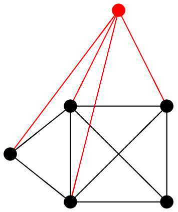
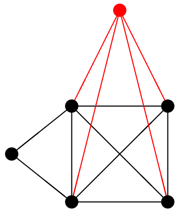


- In the intersection case, this problem can be decided in polynomial time
- Extension is possible if and only if the new vertex extends a clique in the edge clique cover



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- Problem of extending an overlap representation is **NP**-complete
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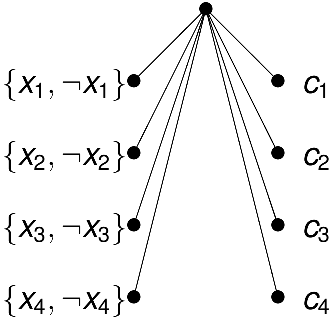
Definition (Not-All-Equal-3SAT)

Instance: A set $\{x_1, x_2, \ldots, x_n\}$ of variables, and a set $\{c_1, c_2, \ldots, c_m\}$ of clauses, each with exactly three literals

Question: Is there a truth assignment to the variables that satisfies at least one, but not all literals of each clause?

Reduction

- Given a set of variables and a set of clauses, we construct the following graph and overlap representation



- This representation can be extended if and only if the satisfiability problem has a solution

Outline

Results

- The overlap number is known for a few simple classes of graphs
- Several problems related to finding the overlap number are **NP**-complete

Open Problems

Open Problem

Is computing the overlap number is **NP***-complete?*

Open Problem

Is there an efficient algorithm to compute the overlap number of a tree?